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# Geometric formulation of mechanical systems subjected to time-dependent one-sided constraints

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**Abstract.** A geometrical setting for time-dependent impulsive constraints is given. This description permits us to treat in a systematic way a wide family of mechanical systems subjected to time-dependent constraints, like moving walls, etc. If the impulsive constraints remain then an 'affine projector' can be defined giving the new initial data in terms of the old ones. Several examples are discussed.

## 1. Introduction

In recent years non-holonomic mechanics has received a definitive boost and it has been incorporated into the so-called geometrical or symplectic mechanics (see [1, 6–8, 15, 17, 22–25, 27, 35–37] and references therein). However, mechanical systems subjected to impulsive constraints have not received much attention although they are widely discussed in classical books [4, 31] (see also [30, 32]). The dynamics of these systems cannot be described by vector fields and a description in terms of implicit differential equations is necessary. Recently [13], a geometric framework has been developed for these kind of systems (in the autonomous case) by extending an implicit description of Lagrangian dynamics due to Tulczyjew [39–41] (see also [5, 18, 28, 29]).

The aim of the present paper is to extend our formalism for mechanical systems subjected to time-dependent impulsive constraints. They constitute an important family of mechanical systems. For instance, the Fermi model [11] studies the acceleration of cosmic rays by momentum transfer from magnetic fields, bouncing particles with a vibrating wall under

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the gravitational field of the Earth which is equivalent to the Fermi model, and its quantum version [3, 38], billiards with moving boundary (see [16], and references therein), and many others. It is remarkable that many of these systems exhibit chaotic behaviour. Other examples are those called piecewise holonomic systems (see [33] and references therein); an extension of our formalism in order to cover these cases would be desirable.

The first point to be remarked is that we need a jet manifold setting to develop the Lagrangian formalism [9, 10, 12, 19]. Indeed, $\pi : E \longrightarrow \mathbb{R}$ is a fibred manifold and the fibre $\pi^{-1}(t)$ is the configuration manifold at time $t$ [14]. The evolution space is then the 1-jet manifold $\pi_1 : J^1\pi \longrightarrow \mathbb{R}$, and the Lagrangian function is $L : J^1\pi \longrightarrow \mathbb{R}$. The equations of motion are then derived by using a cosymplectic structure $(\Omega_L, \eta)$ on $J^1\pi$ where $\Omega_L$ is a 2-form defined from $L$ by using the geometric structure of $J^1\pi$, and $\eta = \pi_1^*(\mathrm{d}t)$ (sections 2 and 3). In [20] two of the present authors have developed an implicit geometric setting for time-dependent mechanics using a symplectic structure on $TJ^1\pi$ defined by the complete lift of $(\Omega_L, \eta)$ (section 3). The Hamiltonian formalism is developed by introducing an adequate fibred manifold $\pi_1^* : J^1\pi^* \longrightarrow \mathbb{R}$ (section 4). The above implicit description is used in the present paper to include in the picture the bundle of Chetaev forces given by the existence of impulsive constraints (section 5). The impulsive constraints are viewed here as functions defining a submanifold $\tilde{C}$ with boundary of $J^1\pi$, say it is locally defined by $\Phi^A = 0, \Psi \geqslant 0$. The vector bundle of constraint forces has no constant rank, indeed the fibres suffer a dimensional jump just at the moment when the impulsive constraints act. In fact, the impulsive forces appear due to the discontinuity of the Lagrange multipliers corresponding to $\Psi$. In order to describe the motions we have to use two curves, one of them giving an account of the jump of momenta (section 6). If some impulsive constraints remain, we construct an affine projector which gives the new initial data in terms of the old ones (section 7). The particular cases of holonomic one-sided and mixed constraints are also considered, and a slight modification of the Chetaev bundle is needed (sections 8 and 9). Several examples are discussed in section 10. Finally, in the appendix we recall the cosymplectic formalism which is used to derive the equations of motion for time-dependent mechanical systems.

## 2. Evolution spaces, vertical endomorphism and second-order differential equations

In this section we will recall some definitions and results concerning the geometry of evolution spaces (for more details see [19, 22, 34]). Let $E$ be an $(n+1)$-dimensional fibred manifold over $\mathbb{R}$, i.e. there exists a surjective submersion $\pi : E \longrightarrow \mathbb{R}$.

We denote by $J^1\pi$ the 1-jet manifold of local sections of $\pi$. An element of $J^1\pi$ will be denoted by $j_t^1\phi$, where $t$ stands for the canonical coordinate on $\mathbb{R}$ and $\phi$ is a local section of $\pi$.

If $(t, q^\kappa)$ are fibred coordinates on $E$, then $J^1\pi$ has local coordinates $(t, q^\kappa, v^\kappa)$. In fact, if $s \longrightarrow \phi(s) = (s, \phi^\kappa(s))$ is a local section of $\pi$ then $j_t^1\phi$ has coordinates

$$\left(t, \phi^\kappa(t), \frac{\mathrm{d}\phi^\kappa}{\mathrm{d}s}(t)\right).$$

Therefore, if $E$ has dimension $(n+1)$, $J^1\pi$ has dimension $(2n+1)$ and it is a fibred manifold over $E$ and $\mathbb{R}$ with canonical projections $\pi_{1,0} : J^1\pi \longrightarrow E$ and $\pi_1 : J^1\pi \longrightarrow \mathbb{R}$, respectively. In local coordinates, we have

$$\pi_{1,0}(t, q^\kappa, v^\kappa) = (t, q^\kappa) \qquad \pi_1(t, q^\kappa, v^\kappa) = t \qquad \pi(t, q^\kappa) = t.$$

Jet manifolds $J^1\pi$ are evolution spaces for time-dependent mechanics.

*Remark 2.1.* If $E = \mathbb{R} \times Q$ and $\pi$ is the trivial fibration $pr_{\mathbb{R}} : E = \mathbb{R} \times Q \longrightarrow \mathbb{R}$, we have the canonical identification $J^1 pr_{\mathbb{R}} = \mathbb{R} \times TQ$, $TQ$ being the tangent bundle of $Q$.

We define a canonical embedding $\iota : J^1\pi \longrightarrow TE$ as follows:

$$\iota(j_t^1\phi) = \dot{\phi}(t)$$

where $\dot{\phi}(t) \in T_{\phi(t)}E$ is the tangent vector at $t$ of the curve $s \longrightarrow \phi(s)$. If we take local coordinates $(t, q^\kappa, \tau, v^\kappa)$ for $TE$, we have

$$\iota(t, q^\kappa, v^\kappa) = (t, q^\kappa, 1, v^\kappa).$$

There exists a canonical (1, 1)-tensor field $\tilde{J}$ on $J^1\pi$ [34] (see also [19, 22]) which is given in local coordinates $(t, q^\kappa, v^\kappa)$ by

$$\tilde{J} = \theta^\kappa \otimes \frac{\partial}{\partial v^\kappa}$$

where $\theta^\kappa = \mathrm{d}q^\kappa - v^\kappa\,\mathrm{d}t$ are the family of local contact forms on $J^1\pi$. $\tilde{J}$ is called the vertical endomorphism of $J^1\pi$.

A vector field $\xi : J^1\pi \longrightarrow TJ^1\pi$ on $J^1\pi$ is said to be a non-autonomous second-order differential equation (NSODE) if

$$\tilde{J}(\xi) = 0 \qquad \eta(\xi) = 1$$

$\eta$ being the 1-form on $J^1\pi$ defined globally by $\eta = \pi_1^*(\mathrm{d}t)$.

Therefore, $\xi$ is a NSODE iff it has the following local expression

$$\xi = \frac{\partial}{\partial t} + v^\kappa \frac{\partial}{\partial q^\kappa} + \xi^\kappa \frac{\partial}{\partial v^\kappa}$$

where $\xi^\kappa = \xi^\kappa(t, q^\chi, v^\chi)$.

A local section $\phi$ of $\pi : E \longrightarrow \mathbb{R}$ is a solution of a NSODE $\xi$ if the 1-jet prolongation $j^1\phi$ of $\phi$ to $J^1\pi$ is an integral curve of $\xi$.

Thus, $\phi(t) = (t, \phi^\kappa(t))$ is a solution of $\xi$ iff it satisfies the following system of non-autonomous differential equations of second order:

$$\frac{\mathrm{d}^2\phi^\kappa}{\mathrm{d}t^2} = \xi^\kappa\left(t, \phi^\chi, \frac{\mathrm{d}\phi^\chi}{\mathrm{d}t}\right).$$

## 3. Lagrangian formalism in jet manifolds

In this section we will recall the geometric formulation of Lagrangian mechanics in jet manifolds.

Let $L : J^1\pi \longrightarrow \mathbb{R}$ be a non-autonomous or time-dependent Lagrangian function. Define the Poincaré–Cartan forms associated to $L$ by

$$\Theta_L = L\eta + \tilde{J}^*(\mathrm{d}L) \qquad \text{(Poincaré–Cartan 1-form)}$$
$$\Omega_L = -\mathrm{d}\Theta_L \qquad \text{(Poincaré–Cartan 2-form).}$$

Denote by $\tilde{p}_\kappa = \partial L/\partial v^\kappa$ the generalized momenta. Then we have

$$\Theta_L = (L - v^\kappa\,\tilde{p}_\kappa)\,\mathrm{d}t + \tilde{p}_\kappa\,\mathrm{d}q^\kappa = L\,\mathrm{d}t + \tilde{p}_\kappa\theta^\kappa. \tag{1}$$

We say that $L$ is regular if the Hessian matrix

$$(W_{\kappa\chi}) = \left(\frac{\partial^2 L}{\partial v^\kappa \partial v^\chi}\right)$$

is non-singular. So, $L$ is regular iff $(\Omega_L, \eta)$ is a cosymplectic structure on $J^1\pi$. This means that $\Omega_L$ and $\eta$ are closed and $\Omega_L^n \wedge \eta$ is a volume form (see the appendix and [10, 19, 26]). In that case, there exists a unique vector field $\xi_L$ on $J^1\pi$ such that

$$i_{\xi_L}\Omega_L = 0 \qquad i_{\xi_L}\eta = 1. \tag{2}$$

Equivalently, if $\flat_L : TJ^1\pi \longrightarrow T^*J^1\pi$ is the vector bundle isomorphism defined by

$$\flat_L(X) = i_X\Omega_L + \eta(X)\eta \tag{3}$$

we have $\xi_L = \flat_L^{-1}(\eta)$. So, $\xi_L$ is the Reeb vector field of the cosymplectic structure $(\Omega_L, \eta)$, and it will be called the Euler–Lagrange vector field.

If $(t, q^\kappa, v^\kappa; \dot{t}, \dot{q}^\kappa, \dot{v}^\kappa)$ and $(t, q^\kappa, v^\kappa; a_t, a_\kappa, b_\kappa)$ are the corresponding coordinates in $TJ^1\pi$ and $T^*J^1\pi$, respectively, then a direct computation, using (1) and (3), shows that

$$\begin{aligned}\flat_L(t, q^\kappa, v^\kappa; \dot{t}, \dot{q}^\kappa, \dot{v}^\kappa) = \Bigg(& t, q^\kappa, v^\kappa; \dot{t} + \dot{q}^\chi \frac{\partial^2 L}{\partial t \partial v^\chi} - \dot{q}^\chi \frac{\partial L}{\partial q^\chi} + \dot{q}^\chi v^\kappa \frac{\partial^2 L}{\partial q^\chi \partial v^\kappa} \\ & + v^\kappa \dot{v}^\chi \frac{\partial^2 L}{\partial v^\kappa \partial v^\chi}, \dot{t}\frac{\partial L}{\partial q^\kappa} - \dot{t}\frac{\partial^2 L}{\partial t \partial v^\kappa} - \dot{t} v^\chi \frac{\partial^2 L}{\partial q^\kappa \partial v^\chi} + \dot{q}^\chi \frac{\partial^2 L}{\partial q^\kappa \partial v^\chi} \\ & - \dot{q}^\chi \frac{\partial^2 L}{\partial q^\chi \partial v^\kappa} - \dot{v}^\chi \frac{\partial^2 L}{\partial v^\kappa \partial v^\chi}, (\dot{q}^\chi - \dot{t} v^\chi)\frac{\partial^2 L}{\partial v^\kappa \partial v^\chi}\Bigg).\end{aligned} \tag{4}$$

This implies that

$$\xi_L = \frac{\partial}{\partial t} + v^\kappa \frac{\partial}{\partial q^\kappa} + \left(-v^\mu \frac{\partial^2 L}{\partial q^\mu \partial v^\kappa} + \frac{\partial L}{\partial q^\kappa} - \frac{\partial^2 L}{\partial t \partial v^\kappa}\right) W^{\kappa\chi} \frac{\partial}{\partial v^\chi}$$

where $(W^{\kappa\chi})$ is the inverse matrix of the Hessian matrix $(W_{\kappa\chi})$.

Therefore, we have (see [19] and [22]) the following.

*Theorem 3.1.* (i) $\xi_L$ is a NSODE.

(ii) The solutions of $\xi_L$ are just the solutions of the Euler–Lagrange equations for $L$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^\kappa}\right) - \frac{\partial L}{\partial q^\kappa} = 0. \tag{5}$$

In what follows, we will give an implicit description of the equations of motion.

Let $\lambda_{J^1\pi}$ be the Liouville 1-form on $T^*J^1\pi$ and $\omega_{J^1\pi} = -\mathrm{d}\lambda_{J^1\pi}$ the canonical symplectic 2-form. Consider on $TJ^1\pi$ the symplectic 2-form $\omega$ given by $\omega = -\flat_L^*(\omega_{J^1\pi})$. Using the results of [20] on complete lifts of cosymplectic structures to tangent bundles, we obtain that

$$\omega = \Omega_L^{\mathrm{c}} + \eta^{\mathrm{c}} \wedge \eta^{\mathrm{v}}$$

where $\Omega_L^{\mathrm{c}}$ is the complete lift to $TJ^1\pi$ of the 2-form $\Omega_L$ and $\eta^{\mathrm{c}}$ (respectively, $\eta^{\mathrm{v}}$) is the complete (respectively, vertical) lift of the 1-form $\eta$.

The symplectic structure $\omega$ is called the complete lift to $TJ^1\pi$ of the cosymplectic structure $(\Omega_L, \eta)$ (see [20]).

Now, denote by $D_\eta^{\mathrm{L}}$ the Lagrangian submanifold of $TJ^1\pi$ defined by

$$D_\eta^{\mathrm{L}} = \flat_L^{-1}(\eta(J^1\pi)).$$

Since $D_\eta^{\mathrm{L}} = \xi_L(J^1\pi)$, we deduce that the local equations defining $D_\eta^{\mathrm{L}}$ are just the Euler–Lagrange equations for $L$. In fact, using (3), we have

$$D_\eta^{\mathrm{L}} = \left\{(t, q^\kappa, v^\kappa; \dot{t}, \dot{q}^\kappa, \dot{v}^\kappa) \in TJ^1\pi \mid \dot{t} = 1, v^\kappa = \dot{q}^\kappa, \dot{t}\frac{\partial^2 L}{\partial t \partial v^\kappa} + \dot{q}^\chi \frac{\partial^2 L}{\partial q^\chi \partial v^\kappa}\right.$$
$$\left. + \dot{v}^\chi \frac{\partial^2 L}{\partial v^\chi \partial v^\kappa} - \frac{\partial L}{\partial q^\kappa} = 0\right\}$$

that is, the local equations defining $D_\eta^{\mathrm{L}}$ are

$$\dot{t} = 1 \qquad v^\kappa = \dot{q}^\kappa \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^\kappa}\right) - \frac{\partial L}{\partial q^\kappa} = 0.$$

The above situation is illustrated by the following commutative diagram:

$$\begin{tikzcd}
TJ^1\pi \arrow[rr, "\flat_L"] \arrow[dr, "\tau_{J^1\pi}"'] & & T^*J^1\pi \arrow[dl, shift left, "\pi_{J^1\pi}"] \\
D_\eta^{\mathrm{L}} \arrow[u] & J^1\pi \arrow[l, "\xi_L"'] \arrow[ur, "\eta"] \arrow[r, "L"] & \mathbb{R}
\end{tikzcd}$$

where $\tau_{J^1\pi} : TJ^1\pi \longrightarrow J^1\pi$ and $\pi_{J^1\pi} : T^*J^1\pi \longrightarrow J^1\pi$ are the canonical projections.

## 4. The Hamiltonian formalism

Let $L : J^1\pi \to \mathbb{R}$ be a regular time-dependent Lagrangian function. We define the map $\mathrm{Leg} : J^1\pi \to T^*E$ by

$$\mathrm{Leg}(j_t^1\phi)(X) = (\Theta_L)_{(j_t^1\phi)}(\tilde{X})$$

for $j_t^1\phi \in J^1\pi$ and $X \in T_{\phi(t)}E$, where $\tilde{X}$ is a tangent vector at $j_t^1\phi$ such that $(T\pi_{1,0})(\tilde{X}) = X$. In local coordinates we obtain

$$\mathrm{Leg}(t, q^\kappa, v^\kappa) = (t, q^\kappa, L - v^\kappa \tilde{p}_\kappa, \tilde{p}_\kappa). \tag{6}$$

Now, if $x$ is a point of $E$ we consider the one-dimensional subspace $(T_v^*E)_x$ of $T_x^*E$ given by

$$(T_v^*E)_x = \{\alpha \in T_x^*E \mid i_u\alpha = 0, \text{ for all } u \in (V\pi)_x\}$$

where $V\pi = \{v \in TE \mid T\pi(v) = 0\}$. Then, the space $T_v^*E = \bigcup_{x\in E}(T_v^*E)_x$ is a vector subbundle of $\pi_E : T^*E \to E$ of rank one. We will denote by $J^1\pi^*$ the quotient bundle $J^1\pi^* = T^*E/T_v^*E$. The vector bundle $J^1\pi^*$ over $E$ has rank $n$ and canonical projection $\pi_{1,0}^* : J^1\pi^* \to E$. It fibres also over $\mathbb{R}$ with projection $\pi_1^* = \pi \circ \pi_{1,0}^* : J^1\pi^* \to \mathbb{R}$.

If $(t, q^\kappa, p_t, p_\kappa)$ are local coordinates on $T^*E$ then we have local coordinates $(t, q^\kappa, p_t)$ on $T_v^*E$ and $(t, q^\kappa, p_\kappa)$ on $J^1\pi^*$.

Let $\nu : T^*E \to J^1\pi^*$ be the canonical projection. We denote by $\mathrm{leg} : J^1\pi \to J^1\pi^*$ the map $\mathrm{leg} = \nu \circ \mathrm{Leg}$. Using (6) and the fact that $L$ is regular, we deduce that Leg is an inmersion and that leg is a local diffeomorphism. Assume, for the sake of simplicity, that $L$ is hyperregular, that is, $\mathrm{leg} : J^1\pi \to J^1\pi^*$ is a global diffeomorphism. In such a case, we define a global section $h : J^1\pi^* \to T^*E$ of the projection $\nu : T^*E \to J^1\pi^*$ by $h = \mathrm{Leg} \circ \mathrm{leg}^{-1}$. (If $L$ is regular we only have local sections of $\nu$.) This $h$ will be called a Hamiltonian section for a reason which will become clear later.

If $\omega_E$ is the canonical symplectic form on $T^*E$, we consider on $J^1\pi^*$ the 2-form $\Omega_h$ given by $\Omega_h = h^*\omega_E$. A direct computation proves that:

(i) $\mathrm{leg}^*\Omega_h = \Omega_L$ and $\mathrm{leg}^*\eta_1 = \eta$, where $\eta_1$ is the 1-form on $J^1\pi^*$ given by $\eta_1 = (\pi_1^*)^*(\mathrm{d}t)$;

(ii) the pair $(\Omega_h, \eta_1)$ is a cosymplectic structure on $J^1\pi^*$;

(iii) if $X_h$ is the Reeb vector field for $(\Omega_h, \eta_1)$, i.e. $i_{X_h}\Omega_h = 0$, $i_{X_h}\eta_1 = 1$, then $\xi_L$ and $X_h$ are leg-related;

(iv) suppose that in local coordinates

$$h(t, q^\kappa, p_\kappa) = (t, q^\kappa, -H(t, q^\kappa, p_\kappa), p_\kappa).$$

Then, the integral curves of $X_h$ satisfy the Hamilton equations

$$\frac{\mathrm{d}q^\kappa}{\mathrm{d}t} = \frac{\partial H}{\partial p_\kappa} \qquad \frac{\mathrm{d}p_\kappa}{\mathrm{d}t} = -\frac{\partial H}{\partial q^\kappa}. \tag{7}$$

As in the Lagrangian formalism, an alternative way to write down the Hamilton equations is the following.

Let $\flat_h : TJ^1\pi^* \longrightarrow T^*J^1\pi^*$ be the vector bundle isomorphism given by

$$\flat_h(X) = i_X\Omega_h + \eta_1(X)\eta_1.$$

Consider on $TJ^1\pi^*$ the symplectic structure $\omega_h$ defined by

$$\omega_h = -\flat_h^*(\omega_{J^1\pi^*}) = \Omega_h^{\mathrm{c}} + \eta_1^{\mathrm{c}} \wedge \eta_1^{\mathrm{v}}$$

that is, the symplectic structure $\omega_h$ is the complete lift to $TJ^1\pi^*$ of the cosymplectic structure $(\Omega_h, \eta_1)$ on $J^1\pi^*$.

Now, denote by $D^h_{\eta_1}$ the Lagrangian submanifold of $TJ^1\pi^*$ defined by $D^h_{\eta_1} = \flat_h^{-1}(\eta_1(J^1\pi^*))$. Since $D^h_{\eta_1} = X_h(J^1\pi^*)$, we obtain that the local equations defining $D^h_{\eta_1}$ are just the Hamilton equations (7).

Finally, using the fact that $(T\,\mathrm{leg})(\xi_L) = X_h$, we conclude that

$$(T\,\mathrm{leg})(D^{\mathrm{L}}_\eta) = D^h_{\eta_1}.$$

The following commutative diagram illustrates the above situation:

$$\begin{tikzcd}
TJ^1\pi \arrow[rr, "\flat_L"] \arrow[ddr, "\tau_{J^1\pi}"'] \arrow[rrrr, bend left, "T\,\mathrm{leg}"] & & T^*J^1\pi \arrow[ddl, "\pi_{J^1\pi}"] & & TJ^1\pi^* \arrow[r, "\flat_h"] \arrow[dd, "\tau_{J^1\pi^*}"] & T^*J^1\pi^* \arrow[ddl, "\pi_{J^1\pi^*}"] \\
 & & & D^h_{\eta_1} \arrow[ur] & & \\
D^{\mathrm{L}}_\eta \arrow[uu] & J^1\pi \arrow[l, "\xi_L"'] \arrow[uur, "\eta"] \arrow[rrr, "\mathrm{leg}"] & & & J^1\pi^* \arrow[ul, "X_h"] \arrow[uur, "\eta_1"] &
\end{tikzcd}$$

## 5. Non-holonomic one-sided constraints

We consider a modification of the Lagrangian formulation of section 3, to include the Chetaev forces due to the presence of non-holonomic constraints.

Let $L : J^1\pi \longrightarrow \mathbb{R}$ be the regular time-dependent Lagrangian function of a mechanical system which is subjected to non-holonomic one-sided constraints determined by a *submanifold* $\tilde{C}$ of $J^1\pi$ *with boundary*, where the boundary $\partial\tilde{C}$ is assumed to be orientable.

A submanifold $N$ with boundary of a differentiable manifold $M$ is understood (see [13]) as a subset $N$ of $M$, locally defined by equations of the form $\Phi^A(x) = 0$, $\Psi(x) \geqslant 0$; so, $N$ is a manifold with boundary in the usual sense. Then, the interior of $N$ (denoted by $\mathrm{Int}\, N$) is a submanifold of $M$ and the boundary $\partial N$ of $N$ is a submanifold of $M$ of codimension one with respect to $N$.

We denote by $T\tilde{C}$ the tangent bundle of $\tilde{C}$, defined as follows. If $x \in \tilde{C}$, then $T_x\tilde{C}$ denotes the tangent vectors $X \in T_x(J^1\pi)$, such that $X(\Phi^A) = 0$, for any $A$. If $x$ is an interior point, then $T_x\tilde{C}$ is just the usual tangent space ($\mathrm{Int}\,\tilde{C}$ is a submanifold of $J^1\pi$). The annihilator $(T\tilde{C})^o$ of $T\tilde{C}$ is locally generated by $\{\mathrm{d}\Phi^A\}$, i.e.

$$(T\tilde{C})^o = \mathrm{span}\,\{\mathrm{d}\Phi^A\}.$$

We consider a distribution contained in $\tilde{J}^*(T^*(J^1\pi))_{|\tilde{C}}$ as follows:

$$(F_1)_x = \begin{cases} \mathrm{span}\ \{\tilde{J}^*(\mathrm{d}\Phi^A)(x)\} & \text{if } x \in \mathrm{Int}\,\tilde{C} \\ \mathrm{span}\ \{\tilde{J}^*(\mathrm{d}\Phi^A)(x),\, \tilde{J}^*(\mathrm{d}\Psi)(x),\, \bar{f}_a(x)\} & \text{if } x \in \partial\tilde{C} \end{cases}$$

where $\bar{f}_a, 1 \leqslant a \leqslant U$, are 1-forms which belong to the image of $\tilde{J}^*$, that is, the local expression of $\bar{f}_a$ is of the form

$$\bar{f}_a = (\bar{f}_a)_\kappa \theta^\kappa.$$

The set $F_1$ is a vector bundle over $\tilde{C}$ in an extended sense since not all the fibres have the same dimension. It represents reaction forces of the constraints (also known as the Chetaev bundle). Here, the 1-forms $\{\bar{f}_1, \ldots, \bar{f}_U\}$ correspond to instantaneous reaction forces at the boundary due to some physical conditions (rough walls, etc).

We assume that there may be, in addition, external forces acting on the system. These forces are introduced as another vector bundle $F_2$ over $\tilde{C}$ which is a vector subbundle of $\tilde{J}^*(T^*(J^1\pi))|_{\tilde{C}}$. We also require that $F_1$ and $F_2$ have a trivial intersection, that is, $F_1 \cap F_2 = 0$, and we consider the Whitney sum $F_1 \oplus F_2$.

We assume that the following two conditions hold:

(i) *admissibility*

$$\dim\,(T\tilde{C})^o = \dim\ \tilde{J}^*(T\tilde{C})^o$$

(ii) *compatibility*

$$\tilde{J}^*(T\tilde{C})^o \cap ((T\tilde{C})^o)^\perp = 0$$

where the orthogonal complement $((T\tilde{C})^o)^\perp$ is defined with respect to the cosymplectic structure $(\Omega_L, \eta)$ on $J^1\pi$ (see the appendix).

*Remark 5.1.* If there are no permanent constraints the compatibility condition is trivially satisfied.

Next, we will give a local interpretation of the admissibility and compatibility conditions. Since

$$(T\tilde{C})^o = \mathrm{span}\ \{\mathrm{d}\Phi^A\} \qquad \tilde{J}^*(T\tilde{C})^o = \mathrm{span}\ \{\tilde{J}^*(\mathrm{d}\Phi^A)\}$$

the equality $\dim\,(T\tilde{C})^o = \dim\ \tilde{J}^*(T\tilde{C})^o$ means that the map

$$\tilde{J}^* : (T\tilde{C})^o \longrightarrow \tilde{J}^*(T\tilde{C})^o$$

is an isomorphism at each point of $\tilde{C}$. Hence, locally the $\{(\partial\Phi^A/\partial v^\kappa)\theta^\kappa\}$ are linearly independent. So, the admissibility condition is locally equivalent to the condition that the reaction forces are independent.

Now, let $\flat_L : TJ^1\pi \longrightarrow T^*J^1\pi$ be the vector bundle isomorphism given by (3) and $\#_L = \flat_L^{-1}$ the inverse homomorphism. We have (see proposition A.2)

$$((T\tilde{C})^o)^\perp = \{\alpha \in T^*(J^1\pi) \mid \langle \#_L(\alpha), (T\tilde{C})^o\rangle = 0\}.$$

Thus $\alpha \in ((T\tilde{C})^o)^\perp$ iff $\langle \#_L(\alpha), \mathrm{d}\Phi^A\rangle = 0$ for any $A$. So, if $\alpha$ belongs to the image of $\tilde{J}^*$ we have that $\alpha = \alpha_\kappa\theta^\kappa$ and, since $\#_L(\theta^\kappa) = -W^{\kappa\chi}\partial/\partial v^\chi$, we deduce that $\alpha \in ((T\tilde{C})^o)^\perp$ iff

$$\alpha_\kappa \frac{\partial\Phi^A}{\partial v^\chi} W^{\kappa\chi} = 0 \qquad \text{for any } A. \tag{8}$$

Therefore, the compatibility condition

$$\tilde{J}^*((T\tilde{C})^o) \cap ((T\tilde{C})^o)^\perp = 0$$

is locally equivalent to the condition that the following matrix is non-singular:

$$\left(W^{\kappa\chi}\frac{\partial\Phi^A}{\partial v^\kappa}\frac{\partial\Phi^B}{\partial v^\chi}\right). \tag{9}$$

Let $\tilde{\flat}_L : TJ^1\pi \times_{J^1\pi} T^*J^1\pi \longrightarrow T^*J^1\pi \times_{J^1\pi} T^*J^1\pi$ be the diffeomorphism defined by

$$\tilde{\flat}_L(X,\alpha) = (\flat_L(X), \alpha). \tag{10}$$

Let us consider the set

$$D = \tilde{\flat}_L^{-1}(\eta(J^1\pi) + (F_1 \oplus F_2), F_1 \cap ((T\tilde{C})^o)^\perp) \subseteq TJ^1\pi \times_{J^1\pi} T^*J^1\pi. \tag{11}$$

Starting with coordinates $(t, q^\kappa, v^\kappa)$ on $J^1\pi$, consider the induced coordinates in $TJ^1\pi$ and $T^*J^1\pi$ as in section 3.

If $(X,\alpha) \in TJ^1\pi \times_{J^1\pi} T^*J^1\pi$, we can locally write

$$X = (t, q^\kappa, v^\kappa; \dot{t}, \dot{q}^\kappa, \dot{v}^\kappa) \qquad \alpha = (t, q^\kappa, v^\kappa; a_t, a_\kappa, b_\kappa).$$

The elements of $F_2$ are locally written as $f_\kappa\theta^\kappa$. Thus, using (4), (10) and (11), we deduce that $(X,\alpha) \in D$ is equivalent to the following

$$\dot{t} = 1 \qquad v^\kappa = \dot{q}^k \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^\kappa}\right) = \frac{\partial L}{\partial q^\kappa} + \lambda_A\frac{\partial\Phi^A}{\partial v^\kappa} + f_\kappa \qquad \text{if } (t, q^\kappa, v^\kappa) \in \operatorname{Int}\tilde{C}$$

$$\dot{t} = 1 \qquad v^\kappa = \dot{q}^k \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^\kappa}\right) = \frac{\partial L}{\partial q^\kappa} + \lambda_A\frac{\partial\Phi^A}{\partial v^\kappa} + \tilde{\mu}\frac{\partial\Psi}{\partial v^\kappa} + f_\kappa + \tilde{v}^a(\bar{f}_a)_\kappa$$

$$\text{if} \quad (t, q^\kappa, v^\kappa) \in \partial\tilde{C}$$

and

$$(t, q^\kappa, v^\kappa; a_t, a_\kappa, b_\kappa) = (t, q^\kappa, \dot{q}^\kappa; -\dot{q}^\kappa a_\kappa, a_\kappa, 0) \in F_1 \cap ((T\tilde{C})^o)^\perp.$$

We conclude that $(X,\alpha) \in D$ is written as

$$\dot{t} = 1 \qquad v^\kappa = \dot{q}^\kappa \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^\kappa}\right) = \frac{\partial L}{\partial q^\kappa} + \lambda_A\frac{\partial\Phi^A}{\partial v^\kappa} + \mu\frac{\partial\Psi}{\partial v^\kappa} + f_\kappa + v^a(\bar{f}_a)_\kappa$$

and

$$(t, q^\kappa, v^\kappa; -v^\kappa a_\kappa, a_\kappa, 0) \in F_1 \cap ((T\tilde{C})^o)^\perp$$

where we let $\mu = \tilde{\mu}$, $v^a = \tilde{v}^a$ on $\partial\tilde{C}$ and $\mu = 0$, $v^a = 0$ on $\operatorname{Int}\tilde{C}$ to unify notation.

Next, we will analyse the condition $\alpha = (t, q^\kappa, \dot{q}^\kappa; -\dot{q}^\kappa a_\kappa, a_\kappa, 0) \in F_1 \cap ((T\tilde{C})^o)^\perp$. We will denote $a_\kappa$ by $\Delta p_\kappa$, from now on. From (8), we deduce that $\alpha = (\Delta p_\kappa)\theta^\kappa \in ((T\tilde{C})^o)^\perp$ if and only if

$$\Delta p_\kappa \frac{\partial \Phi^A}{\partial v^\chi} W^{\kappa\chi} = 0 \qquad \text{for any } A. \tag{12}$$

But $\alpha \in F_1$ means

$$\alpha = (\Delta p_\kappa)\theta^\kappa = \bar{\lambda}_A \tilde{J}^*(\mathrm{d}\Phi^A) + \bar{\mu}\tilde{J}^*(\mathrm{d}\Psi) + \bar{\nu}^a \bar{f}_a = \left(\bar{\lambda}_A \frac{\partial \Phi^A}{\partial v^\kappa} + \bar{\mu}\frac{\partial \Psi}{\partial v^\kappa} + \bar{\nu}^a (\bar{f}_a)_\kappa\right)\theta^\kappa$$

with $\bar{\mu} = \bar{\nu}^a = 0$ on $\operatorname{Int}\tilde{C}$. Hence

$$\Delta p_\kappa = \bar{\lambda}_A \frac{\partial \Phi^A}{\partial v^\kappa} + \bar{\mu}\frac{\partial \Psi}{\partial v^\kappa} + \bar{\nu}^a (\bar{f}_a)_\kappa.$$

Replacing in the conditions (12) to take into account that $\alpha$ belongs to the intersection, we get

$$\bar{\lambda}_A \frac{\partial \Phi^A}{\partial v^\kappa}\frac{\partial \Phi^B}{\partial v^\chi} W^{\kappa\chi} + \bar{\mu}\frac{\partial \Psi}{\partial v^\kappa}\frac{\partial \Phi^B}{\partial v^\chi} W^{\kappa\chi} + \bar{\nu}^a (\bar{f}_a)_\kappa \frac{\partial \Phi^B}{\partial v^\chi} W^{\kappa\chi} = 0. \tag{13}$$

This means that if $\bar{\mu}$ and $\bar{\nu}^a$ are given, we can compute $\bar{\lambda}_A$. In particular, if $\bar{\mu} = 0$ and $\bar{\nu}^a = 0$ we have $\bar{\lambda}_A = 0$ for any $A$ (which is the case in $\operatorname{Int}\tilde{C}$).

## 6. Motions

We will describe the motions for a regular time-dependent Lagrangian system subjected to one-sided non-holonomic constraints as defined in the last section. We will use the notation of section 4.

A *motion* is a curve in $J^1\pi^* \times_E T^*J^1\pi$, i.e. a pair of curves $(\sigma, \varphi)$ where $\sigma$ is a curve in $J^1\pi^*$ and $\varphi$ is a curve in $T^*J^1\pi$ such that $\pi^*_{1,0} \circ \sigma = \pi_{1,0} \circ \pi_{J^1\pi} \circ \varphi = \gamma$, with $\gamma$ a section of $\pi : E \longrightarrow \mathbb{R}$. We assume that the section $\gamma$ is continuous and differentiable from above. The curves $\sigma$ and $\varphi$ are not continuous in general, but possess lateral limits and are differentiable from above. The *jumping curve* $\Delta\sigma$ is defined as follows

$$\Delta\sigma(t) = \bar{\sigma}(t^+) - \varphi(t^+)$$

where $\bar{\sigma}$ is the curve on $T^*J^1\pi$ given by

$$\bar{\sigma} = T^*(\pi_{1,0}) \circ \mathrm{Leg} \circ \mathrm{leg}^{-1} \circ \sigma$$

and

$$T^*(\pi_{1,0}) : T^*_{\gamma(t)}E \longrightarrow T^*_{\pi_{J^1\pi}(\varphi(t))}(J^1\pi)$$

is the pull-back of $\pi_{1,0}$.

The equation of motion is the condition that the image of the curve $(\overbrace{\sigma \circ \mathrm{leg}^{-1}}^{\cdot}, \Delta\sigma)$ is contained in $D$. Thus, if we write

$$\sigma(t) = (t, q^\kappa(t), p_\kappa(t))$$
$$\dot{\sigma}(t) = (t, q^\kappa(t), p_\kappa(t); 1, \dot{q}^\kappa(t), \dot{p}_\kappa(t))$$
$$\Delta\sigma(t) = (t, q^\kappa(t), v^\kappa(t); -v^\kappa \Delta p_\kappa, \Delta p_\kappa, 0)$$

the condition above is equivalent to the following equations

$$v^\kappa = \dot{q}^\kappa \qquad p_\kappa = \frac{\partial L}{\partial v^\kappa} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^\kappa}\right) = \frac{\partial L}{\partial q^\kappa} + \lambda_A \frac{\partial \Phi^A}{\partial v^\kappa} + \mu \frac{\partial \Psi}{\partial v^\kappa} + f_\kappa + v^a (\bar{f}_a)_\kappa$$

$$\Delta p_\kappa = \bar{\lambda}_A \frac{\partial \Phi^A}{\partial v^\kappa} + \bar{\mu} \frac{\partial \Psi}{\partial v^\kappa} + \bar{v}^a (\bar{f}_a)_\kappa$$

the Lagrange multipliers $\lambda_A, \mu, \bar{\lambda}_A, \bar{\mu}$ satisfying the conditions stated in section 5. From these local equations it is clear that $\sigma$ is a curve of momenta, and $\Delta\sigma$ is a curve which at each point gives the jump in momenta produced by the impulsive forces.

## 7. Projection from the constraints

Using the admissibility and compatibility conditions, we have a decomposition

$$T^*(J^1\pi)|_{\tilde{C}} = \tilde{J}^*((T\tilde{C})^o) \oplus ((T\tilde{C})^o)^\perp.$$

So, we can define two complementary projectors

$$\bar{P} : T^*(J^1\pi)_{|\tilde{C}} \longrightarrow ((T\tilde{C})^o)^\perp \qquad \bar{Q} : T^*(J^1\pi)|_{\tilde{C}} \longrightarrow \tilde{J}^*((T\tilde{C})^o).$$

Since $(\Delta p_\kappa)\theta^\kappa \in ((T\tilde{C})^o)^\perp$ we deduce that $\bar{P}((\Delta p_\kappa)\theta^\kappa) = (\Delta p_\kappa)\theta^\kappa$, so that

$$(\Delta p_\kappa)\theta^\kappa = \bar{\mu}\bar{P}\left(\frac{\partial \Psi}{\partial v^\kappa}\theta^\kappa\right) + \bar{v}^a \bar{P}((\bar{f}_a)_\kappa \theta^\kappa).$$

This means that in order to compute the momenta jump we only need to know the Lagrange multipliers $\bar{\mu}$ and $\bar{v}^a$. In some cases, this task can be accomplished in a very geometrical way.

In fact, assume that the impulsive constraints remain after the impulse. More precisely, the 1-forms $\bar{f}_a$ are just given by some impulsive constraints $\Psi^a$ which remain along the boundary of $\tilde{C}$. The reader can imagine, for instance, the motion of a rolling ball on the plane which hits a wall and then it continues the motion rolling also on the wall.

We include, for simplicity, the impulsive constraint $\Psi$ in the set $\{\Psi^a\}$. Our assumption is that the constraints $\Psi^a$ are affine, say $\Psi^a = \Psi^a_\kappa(t,q)v^\kappa + \varphi^a(t,q)$.

In addition, we assume that the Lagrangian $L$ is of mechanical type. Let us explain the meaning of this terminology.

Let $g$ be a metric in the vertical vector bundle $V\pi$ of $\pi : E \longrightarrow \mathbb{R}$, and let $\Gamma$ be a Ehresmann connection in $\pi$. As we know, $\Gamma$ induces a global section $s : E \longrightarrow J^1\pi$ of $\pi_{1,0}$ locally expressed by $s(t, q^\kappa) = (t, q^\kappa, s^\kappa(t,q))$. We define $L : J^1\pi \longrightarrow \mathbb{R}$ by

$$L(j^1_t\phi) = \tfrac{1}{2} g_{\phi(t)}(\mathrm{vert}(\dot{\phi}(t)), \mathrm{vert}(\dot{\phi}(t))) - V \circ \pi_{1,0}$$

where $\mathrm{vert}(\dot{\phi}(t))$ denotes the vertical part of $\dot{\phi}(t)$ with respect to $\Gamma$, and $V : E \longrightarrow \mathbb{R}$ is a potential function. In local coordinates we have

$$L(t, q^\kappa, v^\kappa) = \tfrac{1}{2} g_{\kappa\chi} v^\kappa v^\chi - g_{\kappa\chi} v^\kappa s^\chi + \tfrac{1}{2} g_{\kappa\chi} s^\kappa s^\chi - V(t, q^\kappa) \tag{14}$$

where $g_{\kappa\chi}(t,q) = g(\partial/\partial q^\kappa, \partial/\partial q^\chi)$.

*Remark 7.1.* If $E$ is the trivial bundle $\mathbb{R} \times Q$, $g$ comes from a Riemannian metric on $Q$ and $\Gamma$ is the trivial connection (say, $s^\kappa = 0$), we obtain

$$L(t, q^\kappa, v^\kappa) = \tfrac{1}{2} g_{\kappa\chi}(q) v^\kappa v^\chi - V(t, q^\kappa).$$

If not, one has to assume the existence of an Ehresmann connection in order to have a decomposition of the tangent vectors to $E$ in their horizontal ('$\partial/\partial t$-part') and vertical parts.

The Legendre transformation corresponding to the Lagrangian $L$ given by (14) is

$$\mathrm{leg}(t, q^\kappa, v^\kappa) = (t, q^\kappa, g_{\kappa\chi} v^\chi - g_{\kappa\chi} s^\chi).$$

We assume that the non-holonomic constraints $\Phi^A$ are affine, say $\Phi^A = \Phi^A_\kappa(t, q) v^\kappa + \varphi^A(t, q)$. Then we get $\tilde{J}^*(\mathrm{d}\Phi^A) = \Phi^A_\kappa \theta^\kappa$. Thus, we have

$$\bar{Q}(\alpha) = -\tilde{\mathcal{C}}_{AB}[\#_L(\alpha)(\Phi^B)]\tilde{J}^*(\mathrm{d}\Phi^A) \qquad \bar{P}(\alpha) = \alpha - \bar{Q}(\alpha)$$

for any 1-form $\alpha$, where $(\tilde{\mathcal{C}}_{AB})$ is the inverse matrix of $(\tilde{C}^{AB} = g^{\kappa\chi}\Phi^A_\kappa \Phi^B_\chi)$. Then, we obtain

$$\bar{P}(\tilde{J}^*(\mathrm{d}\Psi^a)) = (\Psi^a_\kappa - \tilde{\mathcal{C}}_{AB}\Phi^A_\kappa \Phi^B_\mu \Psi^a_\chi g^{\chi\mu})\theta^\kappa.$$

For brevity we introduce the notation $\beta^a_\kappa = \Psi^a_\kappa - \tilde{\mathcal{C}}_{AB}\Phi^A_\kappa \Phi^B_\mu \Psi^a_\chi g^{\chi\mu}$.

We will assume that the 1-forms $\{\tilde{J}^*(\mathrm{d}\Phi^A), \tilde{J}^*(\mathrm{d}\Psi^a)\}$ are independent along $\partial\tilde{C}$. Then, the 1-forms $\{\bar{P}(\tilde{J}^*(\mathrm{d}\Psi^a)) = \beta^a_\kappa \theta^\kappa\}$ are also independent along $\partial\tilde{C}$.

Denote by $(v^\kappa)_0$ and $(v^\kappa)_1$ (respectively, $(p_\kappa)_0$ and $(p_\kappa)_1$) the velocities (respectively, momenta) before and after the impulse. Since the impulsive constraints remain we have

$$\Psi^a_\kappa (v^\kappa)_1 + \varphi^a = 0 \qquad \text{for all } a$$

and, moreover,

$$\Phi^A_\kappa (v^\kappa)_1 + \varphi^A = 0 \qquad \text{for all } A.$$

Therefore, we get

$$\beta^a_\kappa (v^\kappa)_1 + \bar{\beta}^a = 0 \qquad \text{for all } a \tag{15}$$

where $\bar{\beta}^a = \varphi^a - \tilde{C}_{AB}\varphi^A \Phi^B_\kappa \Psi^a_\chi g^{\kappa\chi}$. By the Legendre transformation, (15) becomes

$$\beta^a_\kappa g^{\kappa\chi}(p_\chi)_1 + \beta^a \circ s = 0 \qquad \text{for all } a \tag{16}$$

where $\beta^a = \beta^a_\kappa v^\kappa + \bar{\beta}^a$.

Since $\Delta p_\kappa = (p_\kappa)_1 - (p_\kappa)_0 = \sum_a \bar{v}^a \beta^a_\kappa$, from (16) we deduce that

$$(p_\kappa)_1 = (p_\kappa)_0 - \tilde{\mathcal{C}}'_{ab}\beta^a_\kappa \beta^b_\chi g^{\mu\chi}(p_\mu)_0 - (\beta^b \circ s)\tilde{\mathcal{C}}'_{ab}\beta^a_\kappa \tag{17}$$

where $(\tilde{\mathcal{C}}'_{ab})$ is the inverse matrix of the regular matrix $(\tilde{\mathcal{C}}'^{ab} = \beta^a_\kappa \beta^b_\chi g^{\kappa\chi})$.

Equation (17) gives the momentum after the impulse in terms of the momentum before the impulse. We will obtain it by defining a convenient projector.

Let $\tilde{F}_1$ be the vector bundle defined by

$$(\tilde{F}_1)_x = \text{ span } \{\bar{P}(\tilde{J}^*(\mathrm{d}\Psi^a)(x))\} \qquad \text{for all } x \in \partial\tilde{C}.$$

Our purpose now is to define a complementary vector bundle of $\tilde{F}_1$ along $\partial\tilde{C}$ such that the associated projectors give the jump of momenta.

For each $x \in \partial\tilde{C}$, we define

$$\tilde{S}_x = \{\gamma \in T^*_x(J^1\pi) \mid \mathrm{d}\beta^a(x)(\#_L(\gamma)) = 0, \text{ for all } a\}.$$

(We remark that $\tilde{S}_x = \flat_L(\langle \mathrm{d}\beta^a(x)\rangle^o)$).

*Lemma 7.2.* We have

$$T^*_x(J^1\pi) = (\tilde{F}_1)_x \oplus \tilde{S}_x$$

for any $x \in \partial\tilde{C}$.

*Proof.* Since $\{\beta^a_\kappa \theta^\kappa\}$ are independent, then $\{\mathrm{d}\beta^a\}$ also are independent. So, the dimensions of $(\tilde{F}_1)_x$ and $\tilde{S}_x$ are complementary. Now, assume that $\gamma = U_c \beta^c_\kappa \theta^\kappa \in (\tilde{F}_1)_x \cap \tilde{S}_x$. Therefore,

$$0 = (U_c \beta^c_\kappa \#_L(\theta^\kappa))(\beta^a) = -U_c \beta^c_\kappa g^{\kappa\chi} \beta^a_\chi$$

for all $a$, which implies $U_c = 0$, for all $c$. Thus, we get $(\tilde{F}_1)_x \cap \tilde{S}_x = \{0\}$. □

Consider the two complementary projectors

$$\tilde{P} : T^*(J^1\pi)_{|\partial\tilde{C}} \longrightarrow \tilde{S} \qquad \tilde{Q} : T^*(J^1\pi)_{|\partial\tilde{C}} \longrightarrow \tilde{F}_1$$

along the boundary of $\tilde{C}$. A direct computation shows that

$$\tilde{Q}(\gamma) = -\tilde{\mathcal{C}}'_{ab}\, \mathrm{d}\beta^a (\#_L(\gamma)) \beta^b_\kappa \theta^\kappa$$

so that we obtain

$$\tilde{P}((p_\kappa)_1 \theta^\kappa) = ((p_\kappa)_1 - \tilde{\mathcal{C}}'_{ab} \beta^a_\kappa \beta^b_\chi g^{\mu\chi} (p_\mu)_1) \theta^\kappa.$$

We define an affine mapping $\tilde{P}' : T^*(J^1\pi)_{|\partial\tilde{C}} \longrightarrow T^*(J^1\pi)_{|\partial\tilde{C}}$ by putting

$$\tilde{P}'(0_x) = -(\tilde{\mathcal{C}}'_{ab}(\beta^b \circ s)\beta^a_\kappa)\theta^\kappa_x \qquad \text{for all } x \in \partial\tilde{C}$$

and with associated linear mapping $\tilde{P}$. Therefore, we have

$$\tilde{P}'((p_\kappa)_0 \theta^\kappa) = -(\tilde{\mathcal{C}}'_{ab}(\beta^b \circ s)\beta^a_\kappa)\theta^\kappa + \tilde{P}((p_\kappa)_0 \theta^\kappa). \tag{18}$$

However,

$$\tilde{P}((p_\kappa)_0 \theta^\kappa) = \tilde{P}\left((p_\kappa)_1 \theta^\kappa - \sum_a \bar{\nu}^a \beta^a_\kappa \theta^\kappa\right) = \tilde{P}((p_\kappa)_1 \theta^\kappa).$$

Thus, we have

$$\begin{aligned}\tilde{P}'((p_\kappa)_0 \theta^\kappa) &= -(\tilde{\mathcal{C}}'_{ab}(\beta^b \circ s)\beta^a_\kappa)\theta^\kappa + \tilde{P}((p_\kappa)_1 \theta^\kappa) \\ &= -(\tilde{\mathcal{C}}'_{ab}(\beta^b \circ s)\beta^a_\kappa)\theta^\kappa + (p_\kappa)_1 \theta^\kappa - \tilde{\mathcal{C}}'_{ab} \beta^a_\kappa \beta^b_\chi g^{\mu\chi} (p_\mu)_1 \theta^\kappa.\end{aligned}$$

Therefore, from (16), we deduce that

$$\begin{aligned}\tilde{P}'((p_\kappa)_0 \theta^\kappa) &= (p_\kappa)_1 \theta^\kappa - \tilde{\mathcal{C}}'_{ab} \beta^a_\kappa ((\beta^b \circ s) + \beta^b_\chi g^{\chi\mu} (p_\mu)_1) \theta^\kappa \\ &= (p_\kappa)_1 \theta^\kappa.\end{aligned}$$

It should be noticed that the linear part of $\tilde{P}'$ is a projector operator.

Even if the impulsive constraints do not remain, it is still possible, in some cases, to construct similar kinds of operators which give the new initial data in terms of the old ones (see, for instance, examples 10.1 and 10.3, later).

## 8. Holonomic one-sided constraints

In this section, we modify the constructions of section 5 for holonomic one-sided constraints. This is not a particular case of the non-holonomic situation.

Let $C$ be a submanifold of $E$ with boundary, that is, $C$ is locally defined by equations of the form $\phi^A(t,q) = 0$, $\psi(t,q) \geqslant 0$. From $C$ we define the submanifold $\tilde{C}$ of $J^1\pi$ with boundary by the equations

$$\bar{\Phi}^A = (\phi^A \circ \pi_{1,0}) = 0 \qquad \Phi^A = (\phi^A)^c_{|J^1\pi} = 0 \qquad \Psi = (\psi \circ \pi_{1,0}) \geqslant 0$$

where $f^c = \mathrm{d}_T f$ denotes the complete lift to $TE$ of a function $f$ on $E$. These equations mean that $\tilde{C}$ is locally defined as follows

$$\bar{\Phi}^A(t, q^\kappa, v^\kappa) = \phi^A(t, q^\kappa) = 0$$

$$\Phi^A(t, q^\kappa, v^\kappa) = v^\kappa \frac{\partial \phi^A}{\partial q^\kappa} + \frac{\partial \phi^A}{\partial t} = 0$$

$$\Psi(t, q^\kappa, v^\kappa) = \psi(t, q^\kappa) \geqslant 0.$$

In fact, the submanifold $\tilde{C}$ can be defined by

$$\tilde{C} = \{j_t^1\phi \in J^1\pi \mid \phi(t) \in C, \mathrm{d}\phi^A(\phi(t))(\dot{\phi}(t)) = 0 \text{ for any } A\}.$$

In this case, the Chetaev bundle becomes

$$(F_1)_x = \begin{cases} \text{span } \{(\tilde{J}^* \, \mathrm{d}\Phi^A)(x)\} & \text{if } x \in \operatorname{Int} \tilde{C} \\ \text{span } \{(\tilde{J}^* \, \mathrm{d}\Phi^A)(x), (\overline{\mathrm{d}\psi})(x), \bar{f}_a(x)\} & \text{if } x \in \partial \tilde{C} \end{cases}$$

where $\overline{\mathrm{d}\psi}$ is the 1-form given by

$$\overline{\mathrm{d}\psi} = \tilde{J}^*(\mathrm{d}\psi^c_{|J^1\pi}) = \frac{\partial \psi}{\partial q^\kappa}\theta^\kappa.$$

*Remark 8.1.* In order to obtain a projector operator from the impulsive constraints which remain after the impulse, the procedure is similar as in the above case. We only need to consider the permanent constraint $\psi^c_{|J^1\pi} = (\partial\psi/\partial q^\kappa)v^\kappa + (\partial\psi/\partial t) = 0$, apart from the other possible permanent impulsive constraints $\Psi^a(t, q^\kappa, v^\kappa) = 0$.

## 9. Mixed constraints

An interesting situation occurs when we have mixed constraints, that is, the functions $\Phi^A$ are ordinary non-holonomic constraints but we have, in addition, a holonomic one-sided constraint $\psi(t, q) \geqslant 0$. We can again define a submanifold with boundary $\tilde{C}$ of $J^1\pi$ given by the following equations

$$\Phi^A = 0 \qquad \Psi = (\psi \circ \pi_{1,0}) \geqslant 0$$

which is locally written as follows

$$\Phi^A(t, q^\kappa, v^\kappa) = 0 \tag{19}$$

$$\Psi(t, q^\kappa, v^\kappa) = \psi(t, q^\kappa) \geqslant 0 \tag{20}$$

and the Chetaev bundle is given by

$$(F_1)_x = \begin{cases} \text{span } \{(\tilde{J}^* \, \mathrm{d}\Phi^A)(x)\} & \text{if } x \in \operatorname{Int} \tilde{C} \\ \text{span } \{(\tilde{J}^* \, \mathrm{d}\Phi^A)(x), (\overline{\mathrm{d}\psi})(x), \bar{f}_a(x)\} & \text{if } x \in \partial \tilde{C}. \end{cases}$$

*Remark 9.1.* In order to obtain a projector operator from the permanent impulsive constraints, we proceed as in remark 8.1.

## 10. Examples

*Example 10.1.* Consider the case of collision of a free particle in the plane $x, y$ against the moving wall determined by $x^2 + y^2 \leqslant f(t)$ where $f(t) > 0$ for all $t \in \mathbb{R}$. We assume that the particle has mass $m = 1$ and that the trajectory of any individual particle of the wall is contained in a line through the origin of coordinates.

The system is described by:

(i) the regular Lagrangian function

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2)$$

(ii) the one-sided constraint

$$\psi = x^2 + y^2 - f(t) \leqslant 0.$$

The Chetaev bundle $F_1$ is given by

$$F_1 = \begin{cases} \{0\} & \text{if } x^2 + y^2 < f(t) \\ \operatorname{span}\{x(\mathrm{d}x - \dot{x}\,\mathrm{d}t) + y(\mathrm{d}y - \dot{y}\,\mathrm{d}t)\} & \text{if } x^2 + y^2 = f(t). \end{cases}$$

The changes of momenta and velocities are given by

$$\Delta p_x = \Delta\dot{x} = \dot{x}_1 - \dot{x}_0 = \bar{\mu}x$$
$$\Delta p_y = \Delta\dot{y} = \dot{y}_1 - \dot{y}_0 = \bar{\mu}y$$

where $\bar{\mu} = 0$ if $x^2 + y^2 < f(t)$.

(i) Suppose now that the impulsive constraint $\Psi$ remains after the impulsive force acts. Thus, we have

$$2x\dot{x}_1 + 2y\dot{y}_1 - \dot{f}(t) = 0$$

and hence we obtain that

$$\dot{x}_1 = \dot{x}_0 + \frac{\dot{f}(t) - 2x\dot{x}_0 - 2y\dot{y}_0}{2f(t)}x$$
$$\dot{y}_1 = \dot{y}_0 + \frac{\dot{f}(t) - 2x\dot{x}_0 - 2y\dot{y}_0}{2f(t)}y.$$

Using the results in section 7 we can define an affine projector such that

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \end{pmatrix} = \begin{pmatrix} \dot{x}_0 \\ \dot{y}_0 \end{pmatrix} - \frac{1}{f(t)}\begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix}\begin{pmatrix} \dot{x}_0 \\ \dot{y}_0 \end{pmatrix} + \frac{\dot{f}(t)}{2f(t)}\begin{pmatrix} x \\ y \end{pmatrix}.$$

(ii) Suppose that the collision of the particle with the wall is elastic. We need to know the velocity $V$ of a point remaining always in the wall. This velocity is equal to

$$V = \frac{\dot{f}(t)}{2f(t)}(x, y).$$

Denote by $v_1^{\perp}$ and $v_0^{\perp}$ the normal components to the wall of the velocity after and before the collision. Thus, we get

$$2V = v_1^{\perp} + v_0^{\perp}$$

where we assume that the mass $M$ of the wall is $M \gg 1$. After a simple computation, we have that

$$v_0^{\perp} = \frac{x\dot{x}_0 + y\dot{y}_0}{f(t)}(x, y)$$
$$v_1^{\perp} = \frac{x\dot{x}_1 + y\dot{y}_1}{f(t)}(x, y).$$

Then,

$$\dot{f}(t) = x\dot{x}_0 + y\dot{y}_0 + x\dot{x}_1 + y\dot{y}_1.$$

Finally,

$$\dot{x}_1 = \dot{x}_0 + \frac{\dot{f}(t) - 2x\dot{x}_0 - 2y\dot{y}_0}{f(t)} x$$
$$\dot{y}_1 = \dot{y}_0 + \frac{\dot{f}(t) - 2x\dot{x}_0 - 2y\dot{y}_0}{f(t)} y.$$

In this case one can still define a kind of 'projector' which gives the new initial data in terms of the old ones:

$$\begin{aligned}\begin{pmatrix}\dot{x}_1\\ \dot{y}_1\end{pmatrix} &= \begin{pmatrix}\dot{x}_0\\ \dot{y}_0\end{pmatrix} - \frac{1}{f(t)}\begin{pmatrix}x^2 & xy\\ xy & y^2\end{pmatrix}\begin{pmatrix}\dot{x}_0\\ \dot{y}_0\end{pmatrix} - \frac{1}{f(t)}\begin{pmatrix}x^2 & xy\\ xy & y^2\end{pmatrix}\begin{pmatrix}\dot{x}_0\\ \dot{y}_0\end{pmatrix} + \frac{\dot{f}(t)}{f(t)}\begin{pmatrix}x\\ y\end{pmatrix}\\ &= \begin{pmatrix}\dot{x}_0\\ \dot{y}_0\end{pmatrix} - \frac{2}{f(t)}\begin{pmatrix}x^2 & xy\\ xy & y^2\end{pmatrix}\begin{pmatrix}\dot{x}_0\\ \dot{y}_0\end{pmatrix} + \frac{\dot{f}(t)}{f(t)}\begin{pmatrix}x\\ y\end{pmatrix}.\end{aligned}$$

*Example 10.2.* (*The bouncing particle*.) Mechanical systems subjected to time-dependent impulsive constraints are very important physical systems. We mention below two classical examples.

(i) *The Fermi model*. This is a model proposed by Fermi to study the acceleration of cosmic rays by momentum transfer from magnetic fields. The model consists of a particle bouncing between a fixed and an oscillating wall [11]. This system is the prototype of a two-dimensional map with a Smale horseshoe.

(ii) *Bouncing particles with a vibrating wall, under the gravitational field of the Earth*. The classical version of this system is dynamically equivalent to the above Fermi model. The quantum version consists of reflections of ceasium atoms from a vibrating atomic mirror, under the gravitational field of the Earth. These devices produce temporal phase modulation of de Broglie waves and open new possibilities for precision experiments in atom optics, as for example the construction of an atomic Fabry–Pérot interferometer [3, 38].

As both examples are dynamically equivalent we will only discuss the second one. Let us consider the repeated rebounding of a particle under a constant gravitational field with a periodic moving wall $x_W(t) = A\sin\omega t$. The system is described by:

(i) the regular Lagrangian

$$L = \tfrac{1}{2}v^2 - gx$$

(ii) the time-dependent one-sided holonomic constraint

$$\psi = x - A\sin\omega t \geqslant 0.$$

In this case, the Chetaev bundle is

$$F_1 = \begin{cases}\{0\} & \text{if } x - A\sin\omega t > 0\\ \text{span}\,\{\mathrm{d}x - v\,\mathrm{d}t\} & \text{if } x - A\sin\omega t = 0.\end{cases}$$

Thus, if $t_i$ is the instant of time of the $i$th impact of the particle with the moving wall then

$$v_1(t_i) - v_0(t_i) = p_1(t_i) - p_0(t_i) = \mu_{t_i}$$

where $v_1(t_i)$ is the velocity of departure of the particle and $v_0(t_i)$ is the velocity of approach.

The velocity of approach $v_0(t_i)$ verifies that $v_0(t_i) - \dot{x}_W(t_i) < 0$ and the velocity of departure $v_1(t_i) - \dot{x}_W(t_i) \geqslant 0$. Also, we suppose that $v_1(t_i) - \dot{x}_W(t_i) \leqslant \dot{x}_W(t_i) - v_0(t_i)$. Thus, there exists a real number $\alpha$, $0 \leqslant \alpha \leqslant 1$ (the coefficient of restitution), such that

$$v_1(t_i) - \dot{x}_W(t_i) = \alpha(\dot{x}_W(t_i) - v_0(t_i))$$

or

$$v_1(t_i) = (1+\alpha)A\omega\cos(\omega t_i) - \alpha v_0(t_i)$$

where we assume that the mass $M$ of the wall is $M \gg 1$.

Then, the Lagrange multiplier $\mu_{t_i}$ is equal to

$$\mu_{t_i} = (1+\alpha)A\omega\cos(\omega t_i) - (1+\alpha)v_0(t_i).$$

Now, we relate the $i$th impact with the $(i+1)$th impact. Suppose that the distance that the ball travels between both impacts is large compared with the amplitude of the motion of the wall. Then, the time difference between impacts is given by

$$t_{i+1} - t_i = \frac{2v_1(t_i)}{g}$$

and the velocity $v_0(t_{i+1}) = -v_1(t_i)$. Therefore, we obtain that

$$\phi(t_{i+1}) = \phi(t_i) + \frac{2\omega\, v_1(t_i)}{g}$$

$$v_1(t_{i+1}) = \alpha v_1(t_i) + (1+\alpha)A\omega\cos\left(\phi(t_i) + \frac{2\omega\, v_1(t_i)}{g}\right)$$

where $\phi(t) = \omega t$.

Observe that the repeated impacts are modelled by the map

$$f: \quad \delta\tilde{C} \longrightarrow \delta\tilde{C} \quad (t, v) \longmapsto \left(t + \frac{2v}{g}, \alpha v + (1+\alpha)A\omega\cos\left(\omega t + \frac{2\omega\, v}{g}\right)\right).$$

*Example 10.3.* (*A sphere of radius* $r$ *and mass* 1 *rolls without slidding on a horizontal plane*.) At the instant $t$, the sphere hits a moving smooth wall (see [30]). The system is described by:

(i) the regular Lagrangian function

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + k^2(\dot{\theta}^2 + \dot{\varphi}^2 + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta))$$

(ii) the permanent constraints (the sphere rolls without slidding on the plane $z = 0$)

$$\phi_1 = \dot{x} - r\dot{\theta}\sin\psi + r\dot{\varphi}\sin\theta\cos\psi = 0$$
$$\phi_2 = \dot{y} + r\dot{\theta}\cos\psi + r\dot{\varphi}\sin\theta\sin\psi = 0$$
$$\phi_3 = \dot{z} = 0$$

or, after some algebraic manipulations,

$$\phi_1' = \dot{x}\cos\psi + \dot{y}\sin\psi + r\dot{\varphi}\sin\theta = 0$$
$$\phi_2' = \dot{x}\sin\psi - \dot{y}\cos\psi - r\dot{\theta} = 0$$
$$\phi_3' = \dot{z} = 0$$

(iii) the instantaneous constraint on $x - r \geqslant f(t)$

$$\Psi = \dot{x} - \dot{f}(t).$$

The Chetaev bundle $F_1$ is given by

$$F_1 = \begin{cases} \text{span}\,\{\cos\psi\,(\mathrm{d}x - \dot{x}\,\mathrm{d}t) + \sin\psi\,(\mathrm{d}y - \dot{y}\,\mathrm{d}t) & \\ \quad + r\sin\theta\,(\mathrm{d}\varphi - \dot{\varphi}\,\mathrm{d}t), \sin\psi\,(\mathrm{d}x - \dot{x}\,\mathrm{d}t) & \\ \quad - \cos\psi\,(\mathrm{d}y - \dot{y}\,\mathrm{d}t) - r(\mathrm{d}\theta - \dot{\theta}\,\mathrm{d}t), \mathrm{d}z - \dot{z}\,\mathrm{d}t\} & \text{if } x > r + f(t) \\ \text{span}\,\{\cos\psi\,(\mathrm{d}x - \dot{x}\,\mathrm{d}t) + \sin\psi\,(\mathrm{d}y - \dot{y}\,\mathrm{d}t) & \\ \quad + r\sin\theta\,(\mathrm{d}\varphi - \dot{\varphi}\,\mathrm{d}t), \sin\psi\,(\mathrm{d}x - \dot{x}\,\mathrm{d}t) & \\ \quad - \cos\psi\,(\mathrm{d}y - \dot{y}\,\mathrm{d}t) - r(\mathrm{d}\theta - \dot{\theta}\,\mathrm{d}t), \mathrm{d}z - \dot{z}\,\mathrm{d}t, \mathrm{d}x - \dot{x}\,\mathrm{d}t\} & \text{if } x = r + f(t). \end{cases}$$

The relation between the pre-impact and post-impact momenta is obtained from the equations

$$\begin{aligned}
\Delta p_x &= \bar{\lambda}_1 \cos\psi + \bar{\lambda}_2 \sin\psi + \bar{\mu} \\
\Delta p_y &= \bar{\lambda}_1 \sin\psi - \bar{\lambda}_2 \cos\psi \\
\Delta p_z &= \bar{\lambda}_3 \\
\Delta p_\theta &= -r\bar{\lambda}_2 \\
\Delta p_\varphi &= r\bar{\lambda}_1 \sin\theta \\
\Delta p_\psi &= 0
\end{aligned}$$

where $\bar{\mu} = 0$ if $x > r + f(t)$.

As we know, in order to compute the jump in momenta we only need to know the Lagrange multiplier corresponding to the impulsive constraint. Therefore, we determine the Lagrange multipliers $\bar{\lambda}_1$, $\bar{\lambda}_2$ and $\bar{\lambda}_3$ in terms of the multiplier $\bar{\mu}$. Hence, we have

$$\begin{aligned}
\Delta p_x &= \frac{r^2}{k^2 + r^2}\bar{\mu} \\
\Delta p_y &= 0 \\
\Delta p_z &= 0 \\
\Delta p_\theta &= \frac{rk^2 \sin\psi}{k^2 + r^2}\bar{\mu} \\
\Delta p_\varphi &= -\frac{rk^2 \sin\theta \cos\psi}{k^2 + r^2}\bar{\mu} \\
\Delta p_\psi &= 0
\end{aligned}$$

or, in terms of velocities,

$$\begin{aligned}
\Delta \dot{x} &= \dot{x}_1 - \dot{x}_0 = \frac{r^2}{k^2 + r^2}\bar{\mu} \\
\Delta \dot{y} &= \dot{y}_1 - \dot{y}_0 = 0 \\
\Delta \dot{z} &= \dot{z}_1 - \dot{z}_0 = 0 \\
\Delta \dot{\theta} &= \dot{\theta}_1 - \dot{\theta}_0 = \frac{r \sin\psi}{k^2 + r^2}\bar{\mu} \\
\Delta \dot{\varphi} &= \dot{\varphi}_1 - \dot{\varphi}_0 = -\frac{r \cos\psi}{(k^2 + r^2)\sin\theta}\bar{\mu} \\
\Delta \dot{\psi} &= \dot{\psi}_1 - \dot{\psi}_0 = \frac{r \cos\theta \cos\psi}{(k^2 + r^2)\sin\theta}\bar{\mu}.
\end{aligned}$$

In order to obtain a complete description of the post-impact velocities it is necessary to require additional information about the system. For example, assume that the coefficient of restitution is equal to $\alpha$ ($0 \leqslant \alpha \leqslant 1$), then

$$\dot{x}_1 - \dot{f}(t) = -\alpha(\dot{x}_0 - \dot{f}(t)).$$

This last condition determines the Lagrange multiplier $\bar{\mu}$ as a function of the pre-impact velocity, that is,

$$\bar{\mu} = \frac{(1 + \alpha)(\dot{f}(t) - \dot{x}_0)(k^2 + r^2)}{r^2}.$$

Following section 7, we can write in matricial form

$$\begin{pmatrix}(p_x)_1\\(p_y)_1\\(p_z)_1\\(p_\theta)_1\\(p_\varphi)_1\\(p_\psi)_1\end{pmatrix}=\begin{pmatrix}(p_x)_0\\(p_y)_0\\(p_z)_0\\(p_\theta)_0\\(p_\varphi)_0\\(p_\psi)_0\end{pmatrix}-(1+\alpha)\begin{pmatrix}1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\\frac{k^2\sin\psi}{r}&0&0&0&0&0\\-\frac{k^2\sin\theta\cos\psi}{r}&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\begin{pmatrix}(p_x)_0\\(p_y)_0\\(p_z)_0\\(p_\theta)_0\\(p_\varphi)_0\\(p_\psi)_0\end{pmatrix}$$

$$+(1+\alpha)\dot f(t)\begin{pmatrix}1\\0\\0\\\frac{k^2\sin\psi}{r}\\-\frac{k^2\sin\theta\cos\psi}{r}\\0\end{pmatrix}.$$

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## Appendix. Cosymplectic vector spaces and manifolds

Let $V$ be a real vector space of dimension $(2m+1)$, $\eta$ a 1-form on $V$ and $\Omega$ a 2-form on $V$. Then the triple $(V,\Omega,\eta)$ is called a cosymplectic vector space if $\eta\wedge\Omega^m\neq 0$ (see [2, 21, 24]).

If $(V,\Omega,\eta)$ is a cosymplectic vector space and $V^*$ is the dual space of $V$ then the linear map

$$b: V\longrightarrow V^* \qquad v\in V\longrightarrow b(v)=i_v\Omega+(\eta(v))\eta\in V^* \tag{21}$$

is a linear isomorphism. We will denote by # the inverse homomorphism. The vector $\xi=\#(\eta)$ is called the Reeb vector of the cosymplectic vector space $(V,\Omega,\eta)$. It is characterized by the relations $i_\xi\eta=1$ and $i_\xi\Omega=0$.

Let $W$ be a subspace of $(V,\Omega,\eta)$. We define the orthocomplement of $W$ in $V$ with respect to $(\Omega,\eta)$ as the subspace $W^\perp$ given by (see [21, 24])

$$W^\perp=\{v\in V\mid (i_v\Omega-\eta(v)\eta)\in W^o\} \tag{22}$$

$W^o$ being the annihilator subspace of $W$, that is

$$W^o=\{\alpha\in V^*\mid \alpha(w)=0,\ \text{for all } w\in W\}.$$

We have the following.

*Proposition A.1.* If $W$ is a subspace of a cosymplectic vector space $(V, \Omega, \eta)$ then $\dim V = \dim W + \dim W^{\perp}$ and $(W^{\perp})^{\perp} = \{v - 2\eta(v)\xi \mid v \in W\}$. Moreover, if $W \cap W^{\perp} = \{0\}$, we have $V = W \oplus W^{\perp}$.

If $(V, \Omega, \eta)$ is a cosymplectic vector space we can define a 2-vector $\Lambda : V^* \times V^* \longrightarrow \mathbb{R}$ as follows:

$$\Lambda(\alpha, \beta) = \Omega(\#(\alpha), \#(\beta)) \tag{23}$$

for $(\alpha, \beta) \in V^* \times V^*$.

Under the natural identification $V \cong (V^*)^*$ we have that $(V^*, \Lambda, \xi)$ is a cosymplectic vector space with Reeb vector the 1-form $\eta$. Moreover, from (21) and (23), we deduce that

$$\#(\alpha) = -i_\alpha \Lambda + \alpha(\xi)\xi$$

for $\alpha \in V^*$. Thus, using (22), we obtain the following.

*Proposition A.2.* Let $U$ be a subspace of the cosymplectic vector space $(V^*, \Lambda, \xi)$ and denote by $U^o$ the annihilator subspace of $U$, i.e. $U^o = \{v \in V \mid \alpha(v) = 0, \text{ for all } \alpha \in U\}$. Then,

$$U^{\perp} = b(U^o) = \{\alpha \in V^* \mid \#(\alpha) \in U^o\}.$$

Now, suppose that $M$ is a smooth $(2m+1)$-dimensional manifold. $M$ is said to be cosymplectic if a closed 1-form $\eta$ and a closed 2-form $\Omega$ on $M$ exist such that for all $x \in M$ the triple $(T_x M, \Omega_x, \eta_x)$ is a cosymplectic vector space, where $T_x M$ is the tangent space to $M$ at $x$. If $M$ is a cosymplectic manifold then, using the above results on cosymplectic vector spaces, we can define a skew-symmetric tensor field $\Lambda$ of type $(2, 0)$ on $M$. Furthermore, the bracket of functions on $M$ given by

$$\{f, g\} = \Lambda(\mathrm{d}f, \mathrm{d}g)$$

for $f, g \in C^{\infty}(M, \mathbb{R})$, is a Poisson bracket and the symplectic leaves of this Poisson structure are precisely the leaves of the integrable distribution $\ker \eta$ (see [2]). Finally, notice that if $x$ is a point of $M$ and $W$ is a subspace of $T_x M$ (respectively, $T_x^* M$) then we can define the cosymplectic orthocomplement of $W$ in $T_x M$ (respectively, $T_x^* M$).

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